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DECIMAL APPROXIMATIONS

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DECIMAL APPROXIMATIONS

A CHAPTER IN ARITHMETIC

BY

Harry Arnele

~~H.~~ ST. JOHN HUNTER, M.A.

FELLOW OF JESUS COLLEGE, CAMBRIDGE

"There are more things in heaven and earth, Horatio,
Than are dreamt of in your philosophy."—*Hamlet*

London

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AND NEW YORK

1892

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PREFACE.

In the following little book I have worked out, by arithmetical methods, rules for effecting the addition, subtraction, multiplication, and division of decimal quantities, whether non-terminating or not, *accurately* to any assigned decimal (or other) place. This, I believe, has not been accomplished before.

All the old methods which I can find are only roughly approximate, and the results may be true or not. Thus in *Lund's Wood's Algebra* (dated 1841), as an example of addition, the value of e to 5 decimal places is obtained by taking the terms of the series employed one decimal place farther, and where the following figure would be 5 or more, adding 1 to the last figure : the remainder after 6 decimal places is otherwise neglected. Mr. Lund adds—"Each single fraction is calculated to *six* places of decimals, that the figure which occupies the fifth place in their sum may be correct ; and no more terms of the series need be added, because the first five places of decimals are not affected by them." This is simplicity itself. The same *method* is adopted to find e to 7 decimal places in *Greenhill's Differential Calculus* (dated 1891). The results are in both cases correct, but there is no proof whatever ; and they may have been selected on account of the sort of rhythm in the figures, because the same method applied to obtain e to 6 or 8 decimal places fails. See Art. 9.

428397

In the old method of "contracted multiplication," the multiplications are not certainly accurate, and again unity is added where the next figure would be 5 or more; but not otherwise, this being the only allowance made for omissions. Similarly with the old method of "contracted division." Of both these methods I have made what use I could.

This notion of adding 1 in certain cases is still inculcated in the rules for using logarithms, and is founded upon this argument: ·251 (suppose) taken to 1 decimal place is nearer to ·3 than to ·2, but ·24 is nearer to ·2 than to ·3, and, ∴, as the truth is more nearly approached in this way, any ultimate result must be rendered more nearly accurate. It would, by this *method*, follow that, as ·251 is nearest to ·3 to 1 decimal place, ∴ ·251 + ·251 is nearest to ·6, but the argument only proves that the sum is nearer to ·6 than to ·4, which is true; while in fact it is nearer still to ·5.

So, in a simple case of three additions, ·26 + ·26 + ·26, by this method would be exaggerated into ·9 instead of ·78 or ·8 (on this method), whilst ·24 + ·24 + ·24 would only appear as ·6, instead of ·72 or ·7. This is enough to show that the method is not sound.

Where the quantities are non-terminating, *i.e.*, either in commensurable or, if circulating, having a very large and unknown period, the only method possible is one of approximation; but it will be found from the following pages that such approximation can be carried out accurately as far as required. The same methods apply equally well to cases of terminating decimals, and will then be seen to be an abbreviation of the full work which might, of course, be gone through. As to recurring decimals, the most usual rule is to reduce them to simple

fractions, and then proceed, which may sometimes be advantageous, but often is very laborious.

It may be said that logarithms afford an easy method of obtaining accurate numerical approximations. But (1) they cannot touch addition or subtraction ; (2) they are not always entirely reliable ; (3) they cannot give more than seven significant figures, so that their use is strictly limited.

Further, to work a simple question in arithmetic, it would seem absurd to pull out a book of tables. Thus, if, given that in 1851 there were in Great Britain 10,192,723 males and 10,743,749 females, the question be, how many females there were in each hundred of the population, what need of tables if the sum can be worked as below ?

```
20|9|36472 ) 1074|374900 ( 51
             1046|
               28|
```

giving 51 as the answer.

In solving problems involving numerical results, questions often suggest themselves as to how to approximate. I have endeavoured, in the last chapter particularly, to explain such difficulties, which incidentally occur in the methods here given, and lie at the very root of the whole problem.

H. St. J. H.

April, 1892.

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DECIMAL APPROXIMATIONS.

I.

EXPLANATION OF TERMS.

1. By a *row* of figures is intended a row from left to right of the page, such as one of the rows to be added in an ordinary addition sum.

2. By a *column* of figures is meant a row in the direction of from top to bottom of the page, what is often known as a *vertical* row or column ; such as that in which are those figures in an ordinary addition sum, which, at any point of the work, are added together to obtain one of the figures of the sum.

3. By a *bar*, it is convenient to denote a straight line drawn between two consecutive figures in a row, or between two consecutive columns, to separate them.

4. In any number the *significant* figures are all those which commence and end with a digit, not zero.

Ex.—In 0034509800, the figures from 3 to 8, both inclusive, are the significant figures of the number.

5. It is sometimes convenient to use such a symbol as $\dot{6}$, to denote an infinite row of sixes ; and so of any other repeated figures, as $\dot{2}7183\dot{4}$, etc. ; also if $\cdot a$ be used, a is a single digit.

II.

ADDITION.

6. In the addition of two or more decimal quantities, where the sum is only required to be accurate to a given number of places, if the quantities be written down in the usual way, each being taken to the required number of places, the addition can be performed, if only the proper number be *carried* from the remaining figures. This operation has, therefore, to be effected.

Now, if the figures of the *remainder* be considered, *after* some fixed column A, the utmost that they can be is a series of rows of *nines*, and if this be actually the case, these figures may be *represented in the column* succeeding A, by a column of $\dot{9}$'s, which might be replaced by adding 1 to each figure in the column A, and thus simplifying the question. But if these figures after A be less than such column of $\dot{9}$'s, then the sum of *all* the figures after A, if there be n rows of them, is less than $n \times \dot{9}$, *i.e.* less than $10n$, reckoned in the column after A; and therefore the figure to be carried to column A is less than $\frac{10n}{10}$ or n; therefore the figure to be carried to column A is at *most* $n-1$, *i.e.*, in every case *one less* than the *number of rows.* At the same time this number *may* clearly be zero, and therefore its limits are 0 and $n-1$.

Thus the figure to be carried to *any* column from the succeeding figures ranges from 0 to $n-1$, and if column A correspond to the last decimal place to be considered, then 1, 2, 3 *or* more further columns must be written down and added together *until* the addition of the number

$n-1$ to the last figure in their sum can make no difference in the figure to be *carried* to column A. Then the figure so carried to column A must be right, because the proper figure to be carried to the last column *now* considered (being not greater than $n-1$), *if known*, would lead only to the same result as far as column A is concerned.

It will be observed that the addition to the last column considered, of the number $n-1$, tends to be less and less important the greater the number of additional columns considered : thus, according as one or two additional columns are considered, the values of the $n-1$, *in column A*, are respectively $\frac{n-1}{10}$ and $\frac{n-1}{100}$.

It is, therefore, safer in general, even if $n<10$, to consider *two* additional columns at once.

If $n>10$, one additional column would be of no use, while if n be not >10, one additional column might be enough.

The operations indicated must soon terminate, as the importance of $n-1$ in the *remainder* is diminished tenfold with each additional column employed ; unless the remainder when summed takes the form of some figure p carried to column A, followed by a row very nearly equal to a row of nines.

7. In practice the application of the rule above given is generally quite simple.

Ex. 1.—Required to add $129{\cdot}3\dot{5}7\dot{1}$, $22{\cdot}4\dot{2}$, and $109{\cdot}45\dot{2}1\dot{1}$ correctly to 7 decimal places.

Writing down the rows to 9 decimal places, and drawing a bar after the 7th for distinctness, the operation is as follows :—

$$\begin{array}{r|l} 129{\cdot}3571571 & 57 \\ 22{\cdot}4242424 & 24 \\ 109{\cdot}4521121 & 12 \\ \hline 261{\cdot}2335116 & 93 \end{array}$$

Here $n=3$, and adding up the first column after the bar, the sum is 8, and $8+2=10$, which would make a difference in the 7th decimal place. Adding up, then, both additional columns, the sum is 93, and $93+2=95$, $\therefore$ the figure to be really carried is 0, and the sum is easily found $=261{\cdot}2335116$.

Ex. 2.—Add to 5 decimal places $\cdot\dot{3}$, $2{\cdot}3\dot{4}\dot{5}$ and $\cdot0\dot{4}3\dot{2}$. Here

$$\begin{array}{r|l} {\cdot}33333 & 3 \\ 2\,34545 & 4 \\ {\cdot}04324 & 3 \\ \hline 2{\cdot}72203 & 0 \end{array}$$

Here the addition of $n-1=2$ to the 6th decimal column would make no difference in the number 1 to be carried to the 5th place.

Ex. 3.—Find the sum to 3 decimal places of $\cdot32\dot{1}$, $\cdot45\dot{6}7\dot{8}$, $7{\cdot}\dot{3}\dot{2}$ and $5{\cdot}6\dot{2}$.

Here $n-1=3$, and, as in Ex. 2,

$$\begin{array}{r|l} {\cdot}321 & 3 \\ {\cdot}456 & 7 \\ 7{\cdot}323 & 2 \\ 5{\cdot}622 & 2 \\ \hline 13{\cdot}723 & 4 \end{array}$$

and the result is $13{\cdot}723$.

Ex. 4.—Add to 5 decimal places $1{\cdot}543\dot{9}\dot{2}$, $13{\cdot}\dot{7}\dot{2}$, $15{\cdot}5\dot{9}67\dot{3}$, $0{\cdot}\dot{8}935\dot{4}$ and $9{\cdot}13\dot{8}9\dot{2}$.

Here

$$\begin{array}{r|l} 1{\cdot}54392 & 92 \\ 13{\cdot}72727 & 27 \\ 15{\cdot}59673 & 96 \\ 0{\cdot}89354 & 89 \\ 9{\cdot}13892 & 89 \\ \hline 40{\cdot}90041 & 93 \end{array}$$

as $n-1=4$, and the 7th decimal column adds up to 33, and 4 added to this would not alter the 3 carried to the 6th column, which is therefore right as well as the 5th.

In the same way Ex. 1 is right, up to 8 decimal places.

8. A more difficult class of questions arises where it is required to approximate to the sum of an infinite number of arithmetical quantities which continually diminish according to some law, as in obtaining the Naperian logarithms of numbers, or e the symbol denoting their base; or π (the ratio of the area of any circle to the square on its radius), and so on. In such cases, on reducing the successive terms to decimals, the number of rows would be perpetually increasing, and the previous method so far inapplicable. If, however, this reduction to decimals be continued only for some limited number of terms, then account may be taken of the remaining terms by finding some near limit which they cannot exceed, and dealing with this limit as previously with $n-1$. The last term worked should be the last which contains no significant figure before the last decimal place calculated.

Ex. gr. Suppose e required accurately to 7 decimal places, given that $e=2+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\ldots$ *ad inf.*

The remainder of this series after any term as $\frac{1}{\lfloor p}$, may be shown by algebra to be $<\frac{1}{p\times\lfloor p}$ and $>\frac{1}{(p+1)\times\lfloor p}$, so that its limits in any case are easily found.

Thus writing down the successive terms to 9 decimal places until the next term would give no significant figure in the 9th decimal column, the process being continued

farther on to 2 more columns to get e, also to 9 places of decimals :

$2 = 2\cdot$				
$\frac{1}{\lfloor 2} = \cdot 5$				
$\frac{1}{\lfloor 3} = \cdot 1666666$	66			66
$\frac{1}{\lfloor 4} = \cdot 0416666$	66			66
$\frac{1}{\lfloor 5} = \cdot 0083333$	33			33
$\frac{1}{\lfloor 6} = \cdot 0013888$	88			88
$\frac{1}{\lfloor 7} = \cdot 0001984$	12			69
$\frac{1}{\lfloor 8} = \cdot 0000248$	01			58
$\frac{1}{\lfloor 9} = \cdot 0000027$	55			73
$\frac{1}{\lfloor 10} = \cdot 0000002$	75			57
$\frac{1}{\lfloor 11} = \cdot 0000000$	25			05
$\frac{1}{\lfloor 12} = \cdot 0000000$	02			08
$2\cdot 7182818$	23	$\frac{1}{\lfloor 13} = 00$		16
		$\frac{1}{\lfloor 14} = 00$		01
			28	39

For distinctness the 2 pairs of approximation columns are cut off by bars, and

(1) For 7 decimal places there being practically 10 rows, $n-1=9$, and the remainder is less than $\frac{2}{12}$, $\therefore < 1$ in the 9th column, $\therefore$ the result above is right to 7 places, as $23+9+1=33$ would carry no more.

(2) In the same way, $n-1=11$ and the remainder being $<\frac{1}{14}$, $\therefore < 1$, $11+1=12$ would make no difference in the figures in the 8th and 9th decimal colums, which are 28, so that $e=2{\cdot}718281828$ to 9 decimal places.

9. Suppose that, in a decreasing series like the above, after some fixed term, each term = or < *one eleventh* of the preceding term. Then the ratio of the *remainder* after the fixed term to that term is

$$<\left\{\frac{1}{11}+\frac{1}{11^2}+\frac{1}{11^3}+\ldots \text{ to infinity}\right\}:1,$$

i.e., by geometrical progression

$$<\frac{1}{11}\times\frac{1}{1-\frac{1}{11}}:1 \text{ or } <\frac{1}{10}:1 \text{ or } <{\cdot}1:1,$$

which may be convenient for giving a limit to the remainder by inspection. In the same way, if any other number were here in place of 11, as 9, the remainder would be $<\frac{1}{8}$ of the fixed term, and so on.

In e, for example, taking the ratio $\frac{1}{10}:1$, after $\frac{1}{\lfloor 12}$, the remainder $<{\cdot}0000000003$ certainly; after $\frac{1}{\lfloor 14}$, the remainder is certainly $<{\cdot}000000000002$.

A simplification might clearly have been made in calculating e by adding $\frac{1}{\lfloor 3}$ and $\frac{1}{\lfloor 5}$, which are together ·175. Thus in the approximations virtually two working rows are eliminated.

[It may be interesting to test the old method by applying it to find e to 8 decimal places. Then the figures in the 9th decimal column will be

$$7+7+3+9+3+2+6+6+5+2=50.$$

The previous figures are as above, and carrying 5 to the 8th decimal column the sum is 43, etc. Thus the 8th decimal place is given as 3, which is wrong. Similarly, taken to 6 decimal places, the last place is given as 2.]

Where, as in the preceding Art., the last term of the series proceeded to is the last which contains a significant figure up to the last approximation column, and it is clear that the 1st significant figure of the remainder would not come into that column, the simplest rule (in general sufficient) is to count the remainder as another row, and thus add n in the last column to the sum total, to test the addition.

A closer approximation may be obtained in these cases by taking one term *less* of the series, which saves a row, and writing in the last column *the digit*, which is a *superior* limit to the remainder, which will not constitute another row. Thus, in e to 7 decimal places, omitting $\frac{1}{\lfloor 12}$, the rows are reduced to 9, and the remainder, put in the last column, being $< \frac{25}{11}$ is less than 3 *all told;* and the sum of the last 2 columns would be 21, possibly increased by $8+3$ or 11 to 32, instead of 33 as before. So in e to 9 decimal places, the similar sum would be $39-1$ or 38, possibly increased by $10+2$ or 12 to 50, instead of 51. There is also one row less of calculation in this way.

$$\begin{array}{rr} e= & 2{\cdot}666 \\ & +\ {\cdot}041 \\ & +\ {\cdot}008 \\ \text{(remainder)} & +\ {\cdot}0020 \\ \hline & 2{\cdot}71|7 \end{array}$$

For example, for e to 2 decimal places, trying 3, and there being only 3 rows, this is the same with 2 added, and therefore right to 2 decimal places, and $= 2{\cdot}71$. According to the previous method another column would have been wanted.

III.

SUBTRACTION.

10. If the difference of 2 decimal quantities be required, and the result be only wanted to some given decimal place, the sole difficulty will be to know whether *unity* has or has not to be carried from the figures neglected to the row to be subtracted. Suppose the next figure in each quantity to be written down, the one being a and the other b, so that b has to be taken from a. Then three cases arise, when, viz., (1) $a > b$, (2) $a < b$, and (3) $a = b$.

(1) If $a > b$, then whether unity should or should not be *carried* to b, there is a remainder (if only zero) without carrying, *i.e.*, nothing has to be carried in the proposed question.

(2) If $a < b$, unity must be carried to the given decimal place, whether 1 is or is not carried to b.

(3) If $a = b$, 1 will have to be carried or not, according as 1 should or should not be carried to b. Hence the process must be continued on the same principles as in (1) and (2) until this point is settled.

Ex. 1.—From ·989,583 take $\cdot 291\dot{6}$ to 5 decimal places.

Drawing a bar to separate the figures which must be considered, from the remaining figures.

$$\begin{array}{r|l} \cdot 98958 & \dot{3} \\ \cdot 29166 & \dot{6} \\ \hline \cdot 69791 & \end{array}$$

clearly as $6 > 3$, *one* must be carried, and the remainder follows.

Ex. 2.—$6{\cdot}45 - {\cdot}\dot{3}$ to 2 decimal places:

$$\begin{array}{r|l} 6{\cdot}45 & 0 \\ {\cdot}33 & 3 \\ \hline 6{\cdot}11 & \end{array}$$

Ex. 3.—$6{\cdot}4\dot{5} - {\cdot}345$ to 2 decimal places:

$$\begin{array}{r|l} 6{\cdot}45 & 55 \\ {\cdot}34 & 50 \\ \hline 6\ 11 & \end{array}$$

This is a case of (3).

Ex. 4.—$314{\cdot}2\dot{9}0\dot{5} - 180{\cdot}41\dot{6}\dot{2}$ to 7 decimal places:

$$\begin{array}{r|l} 314{\cdot}2905905 & 90 \\ 180{\cdot}4162626 & 26 \\ \hline 133{\cdot}8743279 & \end{array}$$

Ex. 5.—${\cdot}989548\dot{7}65\dot{6} - {\cdot}291448765\dot{7}$ to 5 decimal places:

$$\begin{array}{r|l} {\cdot}98954 & 87656 \\ {\cdot}29144 & 87657 \\ \hline {\cdot}69809 & \end{array}$$

Here 5 extra places have to be examined to show that unity has to be carried.

IV.

MULTIPLICATION.

11. Suppose two decimal quantities to be multiplied together; for example,

$$\begin{array}{r|l} 459{\cdot}63524 & \\ 25{\cdot}4637 & \\ \hline 321 & 744668 \\ 1378 & 90572 \\ 27578 & 1144 \\ 183854 & 096 \\ 2298176 & 20 \\ 9192704 & 8 \\ \hline 11704{\cdot}013 & 860788 \end{array}$$

In the above after the multiplicand has been multiplied by the several figures of the multiplier, the next operation

is one of addition. If, ∴, the result were only required to (say) 3 decimal places, as indicated, and the figures to the left of the bar were all known, the case would be reduced to one of those considered in Chapter II., and it might be necessary to consider 2 further columns, or in fact to *multiply* correctly to 5 decimal places.

Now, on multiplying together any two digits in assigned decimal places, the unit-digit of their product will be in the decimal place corresponding to the sum of the assigned places, as if 6 in the 2nd decimal place of the multiplier above be multiplied by 5 in the 3rd decimal place of the multiplicand, then the unit-digit of their product will stand in the 5th decimal place. If any figure be multiplied by units, the position, relative to the decimal point, of the unit-digit of the product will be unaltered, but if multiplied by 'tens,' the corresponding digit will be raised one place in rank. These considerations, though simple, are often useful. The calculations necessary, however, are avoided by a 'rule of thumb,' constituting the old rule of 'contracted multiplication.' Place the unit-figure of the multiplier under the required place in the multiplicand, *reverse* the positions of all the other figures of the multiplier, and multiply each figure by that immediately above it and then by the figures to the left, placing the unit-results in one column, which will be that corresponding to the place required. Thus, supplying a 'nought,'

```
459·635240
    736452
-----------
919270480|
        0|
        8|
        0|
        9|
        2|
```

Except in the 1st row, only 1 figure in each multiplication is given, to show approximately how the 5th decimal column is obtained, but in *reverse* order. These figures do not all tally with those in the full work, because account has not yet been taken of what may have to be *carried* in multiplying one digit by another, from the multiplication of the figures following the former digit. Thus in multiplying by 4 in the multiplier, $4 \times 2 = 8$, but, really, multiplying the 4 which follows 2 will cause 1 to be carried to the 8, making really 9. In multiplying by 7 it might appear necessary to 'try back' for several figures. Special rules will, however, now be given for settling this difficulty.

12. Beyond a certain bar, whatever the figures in the multiplicand may be, they must amount to less than 1 within the bar, and they cannot exceed a row of 'nines.' Thus, on multiplying such row of 'nines' by 1, 2, 3, ..., 9 respectively, the figure to be carried will be, respectively, 0, 1, 2, ..., 8. Hence the multiplication of *any* row of figures by a digit can at *most* carry *one less* than that digit. This applies equally to any multiplier as 11, 12, etc. This consideration often saves any necessity for examining many figures, Thus, in the example in Art. 11, on multiplying 5 by 7, it is observed that the product by 7 of the figure 9 carries 6, and this is the most that can be carried in such case, ∴ no more figures to the right need be considered.

13. More generally, if some number at least of the figures immediately beyond the bar be known, different cases arise.

(1) If the multiplying digit be **1**, nothing is carried.

(2) For **2**, if the first figure beyond the bar be < 5, then 0 is carried ; otherwise 1 is carried.

(3) For 3, as the figures beyond the bar are not $> \dot{3}$; $> \dot{3}$ but not $> \dot{6}$; or $> \dot{6}$; the figure to be carried is 0, 1, or 2 respectively.

(4) For 4, as the figures are < 25; 25 or more but $< 5\dot{0}$; 5 or more but < 75; *or* 75 or more; the figure to be carried is 0, 1, 2, or 3 respectively.

(5) For 5, as the figures are < 2; 2 or more but < 4; 4 or more but < 6; 6 or more but < 8; *or* 8 or more; the figure to be carried is 0, 1, 2, 3, or 4 respectively.

(6) For 6, as the figures are not $> 1\dot{6}$; $> 1\dot{6}$ but not $> \dot{3}$; $> \dot{3}$ but < 5; 5 or more but not $> \dot{6}$; $> \dot{6}$ but not $> 8\dot{3}$; or $> 8\dot{3}$; the figure to be carried is 0, 1, 2, 3, 4, or 5 respectively.

(7) For 7, as the figures are not $> \dot{1}4285\dot{7}$; $> \dot{1}4258\dot{7}$ but not $> \dot{2}8571\dot{4}$; $> \dot{2}8571\dot{4}$ but not $> \dot{4}2857\dot{1}$; $> \dot{4}2857\dot{1}$ but not $> \dot{5}7142\dot{8}$; $> \dot{5}7142\dot{8}$ but not $> \dot{7}1428\dot{5}$; $> \dot{7}1428\dot{5}$ but not $> \dot{8}5714\dot{2}$; *or* $> \dot{8}5714\dot{2}$; the figure to be carried is 0, 1, 2, 3, 4, 5, or 6 respectively.

(8) For 8, as the figures are < 125; not < 125 but < 25; not < 25 but < 375; not < 375 but $< 5\dot{0}$; not $< 5\dot{0}$ but < 625; not < 625 but $< 75\dot{0}$; not $< 75\dot{0}$ but < 875; *or* not < 875; the figure to be carried is 0, 1, 2, 3, 4, 5, 6, or 7 respectively.

(9) For 9, as the figures lie between 0 and $\dot{1}$; $\dot{1}$ and $\dot{2}$; $\dot{2}$ and $\dot{3}$, and so on; the figure to be carried is 0, 1, 2, and so on.

As $9 \times \dot{4} = 4 \times \dot{9}$, and so on, the case of 9 can always be reduced to the principle of Art. 12. These results may be used sometimes for reference. They are easily obtained, from the consideration that the division by any given digit a of *some number* ($< a$) of 'tens' must give the

'critical' value of the figures in the multiplicand, for which *such number* will just be, or not be, carried, as appears from multiplying such 'critical' value by a.

Thus in (7) above, the 'critical' values are only obtained by expressing as decimals $\frac{10}{7}$, $\frac{20}{7}$, $\frac{30}{7}$, and so on; for if it be required to find whether a certain row of figures would, on multiplication by 7, carry 2 suppose, or 1, to the next higher decimal place, effecting the division $\frac{20}{7}$, the figures are obtained which the known row must respectively exceed or not.

14. The rows commenced in Art. 11 may now be written down correctly, thus :—

$$\begin{array}{r|l}
459{\cdot}635240 & \\
736452 & \\
\hline
919270480 & \\
229817620 & \\
18385409 & \\
2757811 & \\
137890 & \\
32174 & \\
\hline
11704{\cdot}013 & 84
\end{array}$$

Here 4 in the multiplier multiplied by 4 carries 1 (and 1 *at most*) to 4×2 making 9; $6\times2=12$, and 5 more $(=6-1)$ carried to this would still only carry 1 to 6×5, making 31; so 3×5 even with 2 more carries only 1 to 3×3, making 10; 7 has been considered before in Art. 12, which in this particular example applies to every case. The next column will contain only 4 rows, the first two ending, and $84+3=87$, which makes no difference

in the fourth decimal place even, so that the product true to 4 decimal places has thus been obtained.

15. If in the above example the multiplicand had been chosen for multiplier, the first two lines would have been written

```
25·4637000
  42536954,
```

and though there are here 8 multiplying figures, 4 of the rows would contain zero in the 6th decimal column, so that the amount $n-1$ would be the same as before. As the smaller this amount is, the more easy it generally is to approximate for the additions, when the columns do not end it is generally advisable to select for the multiplier that quantity which has the fewer multiplying figures, as n cannot exceed the number of these actually used for any particular column.

16. Some ordinary examples may now be given.

Ex. 1.—Multiply 27·14986 by 92·41035 to 4 places of decimals. Proceeding to 5 places

```
 27·149860
   5301429
 ---------
 244348740|
    5429972|0
    1085994|
      27149|
        814|
        135|
 ---------
2508·92804|
```

There are only 4 rows in the next column, and 3 added to the last figure 4 in the above does not affect the 4th decimal figure, ∴ the result is 2508·9280.

Ex. 2. —Find $245{\cdot}378263 \times 72{\cdot}4385$ to 5 decimal places. Trying to 6 places,

$$\begin{array}{r|l}
245{\cdot}3782630 & \\
583427 & \\
\hline
17176478410 & \\
490756526 & 0 \\
98151305 & \\
7361347 & \\
1963026 & \\
122689 & \\
\hline
17774{\cdot}833303 &
\end{array}$$

Here there are only 4 rows in the next column, ∴ the result is obtained true to 5 places, viz., 17774·83330. In multiplying here by 8, $8 \times 2{\cdot}6 > 8 \times 2{\cdot}5 > 20$, ∴ 2 is carried; since $8 \times 2 \ldots < 8 \times 3 < 24$, ∴ < 30.

Ex. 3.—Find to 5 decimal places $\cdot 248264 \times \cdot 725234$. In Exs. 1 and 2, from zero being added to the multiplicand, it could easily be seen that the columns in the 6th and 7th decimal place respectively would only have 4 rows, ∴ one additional column was tried; but here there would be 6 rows in the 7th decimal column, ∴, proceeding to 7 places,

$$\begin{array}{r|l}
\cdot 248264 & \\
432527 & \\
\hline
1737848 & 0 \\
49652 & \\
12413 & \\
496 & \\
74 & \\
9 & \\
\hline
\cdot 1800492 &
\end{array}$$

Here there is no unit-figure in the multiplier, but of course the figures are placed as they would be if there were one. Now only 4 can be carried to the last figure 2,

$\therefore$ the product $= \cdot 180049$ accurately to one more decimal place than was required, and $\therefore$ also to 5 decimal places.

Ex. 4.—Find $3670 \cdot 257 \times 12 \cdot 61158$ to 2 decimal places. Here to 3 decimal places,

```
 12·611580
  7520763
 ---------
 37834740|0
  7566948|0
   882810|60
     2522|3
      630|5
       88|2
 ---------
46287·739|6
```

The 3rd decimal column added up comes to 18, the figure to be carried from which would be altered by adding 3. Hence the next column must be calculated, in finding the last figure of which 0 is carried to 7×6, because $11\ldots < 12$, $\therefore 7 \times 11\ldots < 100$. Also the 2nd column beyond the bar will have but 3 rows, $\therefore$ the total is correct to 3 decimal places as well as 2. [This example is given in *Todhunter's Trigonometry*, in which it is worked by logarithms, without any apparent advantage, in their use, over the above work, which moreover gives the result accurately and to 1 place farther.]

In Exs. 3 and 4 above it may be observed that digits recur in the multiplier. In such cases, as the full multiplication by them would produce exactly the same figures, the results must agree as far as they go, so that the first having been obtained the other (or others) can be written down from it.

17. One class of cases remains for consideration, viz., where the multiplier contains many decimal places as well as the multiplicand.

Suppose it required to find to 2 decimal places

$$4{\cdot}5963254 \times 25{\cdot}4317689.$$

Then proceeding to 4 decimal places, and omitting the rows obtained until 6 is reached in the multiplier,

$$\begin{array}{r|l} 0004{\cdot}5963254 & \\ 9867\ 13452 & \\ \hline \cdot & \\ \cdot & \\ 2 & 75 \\ & 36 \\ & 04 \end{array}$$

Thus the 6 carries 2 to the 4th decimal column, and the row to which it belongs is accounted for by the previous method; but writing down a few figures corresponding to 8 and 9 (the 4 corresponding to 9 arising because $9 \times 45\ldots > 400\ldots$), it appears that there are two further rows ('sliding rows'), having no figure within the bar. There would clearly be more of these rows if there were more decimal places in the multiplier, and an ∞ number if there were an ∞ number of decimal places.

Now, the first figure of the multiplier (here 6) which is under a cipher prefixed to the multiplicand, must be worked with to find if it carries a figure within the bar; but if the remaining figures (here 8 and 9) be as great as possible, *i.e.*, a row of 'nines,' they would be < 1 added to such first figure. Hence their effect is less than if *the multiplicand* be *written over again*, beginning *in the first column after the bar*, and $\therefore$ equivalent only to a proper fraction in the 4th decimal column; $\therefore$, if in the additions n be tried instead of $n - 1$, the operation would be safe, since $(n - 1 + \text{fraction}) + \text{fraction}$ cannot be

$$> n + \text{fraction}.$$

The figures in a decimal quantity, it may be noticed, are, if unlimited in number, either circulating or not. In the first case as many of them may be written down as are required, if the circulating figures be known; but in the second case, when the quantity is incommensurable, only some certain number of figures will generally be known, and if more be wanted they must be worked out, as in the case of e before.

18. An example may now be introduced of incommensurable quantities.

Suppose it required to find $\pi \times \sqrt{2}$ to 3 decimal places.

Proceeding to 5 decimal places taking $\pi = 3{\cdot}14159\ldots$, and obtaining $\sqrt{2} = 1{\cdot}414213\ldots$.

```
03·14159
 3124141
 ------|
  314159|
  125663|
    3141|
    1256|
      62|
       3|
       0|
        |314...
 ------|
 4·442|84|
 ------|
```

Here the radical is obtained to 6 decimal places to see if any figure is carried within the bar; also the unknown figures of the multiplier produce less than 314... after the bar, and counting 7 rows, by Art. 17, to 84 must be added 7, making 91. Thus the result is *accurately* 4·442, although the decimal places in π and $\sqrt{2}$ are both infinite in number.

In this case the principle in Art. 16 applies. Also more decimal places in π might have been required, as if the unit of the multiplier had been 2 or more.

The general question of how far to approximate is treated more fully in Chap. VI.

If the last figure, 3, obtained in $\sqrt{2}$ above, were not known, by reasoning similar to that in Art. 17, the remaining rows would amount to less than 3·14... *within* the bar, so that 3 would take the place of 0 in the column before the bar, and $7-1+3=9$ added to 84 would leave the result still correct to 3 decimal places.

V.

DIVISION.

19. In the division of decimals, when the result is only required to a certain number of decimal places, the ordinary rules enable an abbreviation to be made in the dividend, if that be non-terminating or contain many figures, and the divisor contain only a finite number of known figures. For, clearing the divisor of decimals, and placing the decimal point in the dividend accordingly, only so many decimal figures of the latter will then be required as equals the number of them wanted in the quotient ; and the succeeding figures in the dividend may be neglected.

But in *general* a further abbreviation may be made in the divisor, and at each step of the division. In ordinary *long division* the first figure of the quotient may usually be obtained from the 1st figure of the divisor, and the 1st *or* 1st two figures of the dividend, allowing for what must be carried from the multiplication of the other figures of the divisor. This will be safe if the remainder of the multiplication row do not exceed the corresponding

figures in the dividend, or even if it do, and ∴ 1 more has to be carried on to the multiplication of the 1st figure in the divisor, *if* there be room for such carried figure.

Now, if this first division were carried on so as to include a few more figures of the divisor, and the corresponding figures of the 1st remainder obtained; in a similar manner the 2nd figure of the quotient could usually be obtained from such 1st remainder thus cut short, shortening the divisor by the last figure before employed, and so on, until only 1 or 2 figures appeared in the remainder from which to get, by dividing by the 1st figure of the divisor alone, the final figure required in quotient. This would clearly amount to 'barring' off the rest of the full work, and as it would give as many figures in the quotient as the number of divisions effected, it would involve using an equal number at least of figures in the divisor at the first step. Consequently, the same number or 1 more would be wanted in the dividend.

In most approximate questions on division, the result is required to some number of decimal places, and the number of figures before the decimal point in the quotient is so far unknown. If, however, the decimal point be placed in the divisor immediately after its 1st figure, and that of the dividend in accordance, the number of decimal places to be taken in the dividend then equals the number required in the quotient. So that the dividend can be written down as far as it will be wanted, and such a number of figures taken in the divisor as to divide up to the end of the dividend so limited, at the 1st division.

This will appear much more readily from an example. *Suppose* a sum of £25,089,281 to be equally divided

amongst 924,103 persons (or otherwise) how many pounds would each receive, neglecting decimals of £1?

Here 0 decimal places are required, and ∴ pointing as above, the dividend becomes 250 simply, and ∴, to divide this through at the 1st step, the divisor is taken as 9·2. Thus,

```
9|·2 ) 250 | ( 27
       184 |
       ---
        66 |
        64 |
        --
         2 |
```

In multiplying account is taken of the omitted figures of the divisor, and as allowance has to be made on each subtraction, for 1, which *may* be carried from the remaining figures, altogether 2 may require to be allowed for: this is provided for in the remainder 2.

At the second division 66 is divided only by 9, the 2 in the divisor being barred off. Hence the result is £27.

20. For a more extensive example, suppose it required to obtain, to 4 decimal places,

$$2508{\cdot}928065051 \div 92{\cdot}41035.$$

Here, pointing the divisor after the 1st figure, it appears that 7 figures of the dividend are necessary; and ∴ to divide, at the 1st step, 6 figures of the divisor are required. [By the ordinary method of division, even to 4 decimal places, every figure given would have been wanted.] The next figure in the divisor is also written down to carry in multiplying. [If the 1st figure of the quotient were one, this would here be unnecessary; in other cases *more* figures *might* be wanted.]

```
Thus 9·|2|4|1|0|3 (5))250·8928 ( 27·1498
                      184 8207
                      ---------
                       660721 | ?−1
                       646872
                       ------
                        13849 | ?−2
                         9241
                        -----
                         4608 | ?−3
                         3696
                         ----
                          912 | ?−4
                          831
                          ---
                           81 | ?−5
                           73
                           --
                            8 | ?−6
```

The possible allowances to be made for carrying at the subtractions are indicated by -1, etc., to the right.

The second division is made by the 1st 5 figures of the divisor, the 3 being barred off, and so on. The last remainder appears as 8, which *may* have to be diminished by as much as 6, and ∴ so far as yet known ranges from 8 to 2, but in any case there is a *positive* remainder, which is not too great, and is of course the 1st figure in the *next* row of the complete division.

21. It is clear that the number thus allowed for what should be carried at the subtractions equals the number of divisions effected, and if this number be $>$ than the first figure of the divisor, then in the last division this would make a possible difference of 1 at least in the last figure of the quotient. In such cases, ∴, the divisor and dividend must be started with one more figure each, as if one more figure were wanted in the quotient, the last one obtained being unreliable. This may prove necessary in any case. In fact, this difficulty might arise at any stage, as if there were not margin enough for the possible figures

to be carried, so far, from the subtractions. *Cf.* further Art. 23.

If in the example in the previous Art. the division be effected in full, this work may be compared with the abbreviation, and it will be found that 1 has in fact to be subtracted severally from the 2nd, 5th, and 6th remainders, leaving 5 instead of 8 in the last.

If it happened that the figures in the divisor were too few for the above process (not because they were unknown, but because there were no more), noughts could be added, or, what would amount to the same thing, the division could be effected accurately until the last included figure of the dividend could be worked up to at one division, and then proceeded with as above. In this case nothing would be carried at the first subtraction from beyond the bar, because the division up to the bar would be correct. This is indicated hereinafter by 'right,' written instead of '-1.'

22. One or two examples may now be given of the general method.

Ex. 1.—Find $721{\cdot}17562 \div 2{\cdot}257432$ to 3 decimal places.

Here, as the first figure 2 of the divisor is clearly less than the number of divisions required, ∴ (Art. 21) proceeding to 4 decimal places, all the divisor being wanted,

```
2·2|5|7|4|3|2 )721·1756 ( 319·467
               6772296
               -------
                439460 | right
                225743
                ------
                213717 | ?−1
                203168
                ------
                 10549 | ?−2
                  9029
                 -----
                  1520 | ?−3
                  1354
                  ----
                   166 | ?−4
                   158
                   ---
                     8 | ?−5
```

In this, the first division being correct, the 1st subtraction is right; in multiplying by 9, as $32 < 33$, only 2 are carried; in multiplying by 7, as $\frac{40}{7} = 5{\cdot}71\ldots$, which $< 5{\cdot}74$, $\therefore$ 4 has to be carried. Also the last remainder ranges from 8 to 3, which are both positive, and < 22. Hence the result is correct to 3 decimal places.

Ex. 2.—Find $12{\cdot}169825 \div 3{\cdot}14159$ to 5 places of decimals.

Proceeding here to 6 decimal places (Art. 21), one more figure is required in the divisor, thus dividing twice by the divisor in full,

```
3·1|4|1|5|9 )12·169825 ( 3·87377
             942477
             -------
            2745055  right
            2513272
            -------
             231783  right
             219911
             ------
              11872  ?−1
               9424
              -----
               2448  ?−2
               2199
              -----
                249  ?−3
                219
               ----
                 30  ?−4
```

Here the 7's are easily worked, also the 3's; and the last remainder ranging from 30 to $26 < 31$ the last divisor.

Ex. 3.—In working division by logarithms only 7 significant figures can be obtained, and the abbreviated work in the following example, from *Todhunter's Trigonometry*, may be compared for accuracy and labour with the solution therein by logarithms.

To divide ·1234567 by 54·87645. As 5, the 1st figure of

the divisor, < 7, for 7 figures, dividing twice by the divisor in full (Art. 21), there being only 7 figures in the divisor,

```
5|4|·8|7|6|4|5 )·123456700|( ·0022497209
                10975290  |
                ---------
                 13703800 |
                 10975290 |
                 --------
                  2728510 |
                  2195058 |(0)
                  -------
                   533452 |0 right
                   493888 |
                   ------
                    39563 |∴ −1
                    38413 |
                    -----
                     1150 |?−1
                     1097 |
                     ----
                       53 |?−2
                        0 |
                       --
                       53 |?−2
                       49 |
                       --
                        4 |?−3
```

Trying the first figure 5 to divide by alone, an 8th significant figure is obtained in the quotient. The 3rd remainder must have 0 beyond the bar, ∴ 1 must be carried from the 4th remainder, and this leaves the last remainder = from 4 to 1.

Todhunter gives for the result ·002249722, which is not quite right.

This working might seem to contradict Art. 21, but here 4 of the 8 possible subtractions are accounted for, leaving only 4 which < 5.

23. To take one or two of the quasi-failing cases of the method, suppose (1) $55{\cdot}44621 \div 92{\cdot}41035$ required to 1 decimal place, then

```
9(2))5·5|4( ·60
      55|4
       0|0
```

and ·6 may be too much, but if the work be continued 1 place beyond the bar, the result is still the same. This indicates that the quotient is either ·6 or very nearly so, as ·59.... In a finite case like this the shortest plan would probably be to multiply the divisor mentally by ·6, and see if anywhere the product exceeded the dividend. In fact, ·6 is here exact.

Suppose again 113·6647305÷92·41035 wanted to 2 decimal places, then

```
9|2·4(1))113·6|6( 1·230
        924|1
         212|5 right
         184|8
          28|7 carry 1
          27|7
           1|0
```

thus the 3 in the quotient is doubtful on account of what may have to be carried, and proceeding 1 place beyond the bar the next figures certainly carry 1 and leave 0, subject to further allowance. Thus the result is either 1·23 or very nearly so, as 1·229.... It would be necessary to multiply the divisor right out, or else continue the approximations to make certain. The quotient is 1·23, but obviously a small decrease in the tail of the dividend would have reduced this to 1·229....

If the divisor and quotient were really non-terminating, such multiplication of them would seldom agree with the dividend even to the last figure chosen in the latter. Thus in the example in Art. 20, if the next figures in the

quotient were $> \dot{1}$, their multiplication by 9 would raise another unit into the 4th decimal place of the dividend. In fact the product of the divisor by the partial quotient, to the decimal place assigned, falls short of the partial dividend by exactly the remainder, (in this case 8). So in Ex. 2 of Art. 22, the similar product would not agree with the dividend beyond 3 decimal places.

24. *Second Method for the Subtractions.*

There is another method of dealing with the figures to be possibly carried from the subtractions. Instead of leaving them at their maximum number, it is often possible to get this within narrower limits by extending the multiplication rows one step (or more) beyond the bar.

Thus, taking Art. 22, Ex. 1, to 3 decimal places,

```
2|·2|5|7|4|3(2)721·175|6( 319·467
              677229|60
              ------|--
               43946|0 right
               22574|3
               -----|-
               21372|
               20316|8
               -----|-
                1056|
                 902|9
                ----|-
                 154|
                 135|4
                 ---|-
                  19|
                  15|8
                  --|-
                   4|
```

Now the remainder to the right of the bar below the last line here drawn, and following 4, the last partial remainder only arises from subtracting *all* the multiplication rows above it (which begin with 6, 3, 8, 9, 4 and 8) *from*

the remainder of the dividend to the right of the bar. Also the sum of these rows *estimated in the column beyond the bar* is greater than $6+3+8+9+4+8$ or 38, and is less than $38+5=43$ (formed by adding unity to each of the previous figures except the first, as that has no continuation). Hence to subtract from the dividend of 6, 40 *must* be borrowed in *either* case; *i.e.*, the figure to be carried for the subtractions from the last remainder 4 is also 4. This can be done, and therefore the result is correct, the division having been completely worked up to the bar.

It may be seen that this method is nearly the same as the first, of approximating a step further, but it is perhaps neater and less laborious.

```
Similarly   7·|3|5|2|4 )8·6134 0( 1·1715
                        73524 0
                        -----
                        12610|
                         7352|4
                        -----
                         5258|
                         5146|6
                        -----
                          112|
                           73|5
                          ---
                           39|
                           36|7
                           --
                            3|
```

Here the last remainder 3 might be too small by unity, but $0+4+6+5+7=22$, and $22+4=26$, and in either case only 30 is wanted to borrow, *i.e.*, only 3 is to be taken from the remainder, and therefore the result is correct.

VI.

RULES FOR AND EXAMPLES OF APPROXIMATION.

25. In the abbreviated methods of *addition* and *subtraction* the degree of accuracy required in the figures to be dealt with is sufficiently indicated in the methods to be employed.

Similarly in the abbreviated method for multiplication. Multiplication by a single digit *may*, as appears from Art. 13, involve several figures beyond the decimal point required in the multiplicand. *In general*, however, from consideration of such an example as that in Art. 18, if one extra figure only in the multiplicand be allowed for carrying, it may be seen that there will be required in the multiplicand beyond the number of decimal places to be 'tried' for, a number = the number of whole figures in the multiplier, and correspondingly a number of decimal places in the multiplier = the number of whole figures in the multiplicand + the number of decimal places to be tried for. This is a mode of formulating the rule; in practice the numbers of decimal places appear from the rule for writing the figures down.

In complex cases where more than one operation has to be performed, the successive approximations must be carried out with a view to that in the final operation.

Thus, required to find $4{\cdot}002 \times 608{\cdot}27 \times {\cdot}0425839$ correctly to 2 places of decimals.

If the 3rd quantity be multiplied to 4 places of decimals by the product of the other two, and the decimal part of *their* product be *abc* ..., then without writing down the *whole* figures, the ordinary rule would give

$$\begin{array}{r} \cdot 0425839 \\ cba\ldots, \\ \hline \end{array}$$

so that the product of the 1st two quantities should be right to 3 decimal places (instead of 4 as in the rule in this Art., because the 1st figure in the multiplicand here is 0); thus to 4 decimal figures, there being only 2 rows,

$$\begin{array}{r|l} 608\cdot 2700 & \\ 2004 & \\ \hline 2433\cdot 080 & 0 \\ 1216 & 5 \\ \hline 2434\cdot 296 & 5 \\ \hline \end{array}$$

(1 more to 5 would be only 6), and

$$\therefore \quad \begin{array}{r|l} \cdot 0425839 & \\ 6924342 & \\ \hline 85\cdot 16 & 78 \\ 1703 & 35 \\ 127 & 75 \\ 17 & 03 \\ & 85 \\ & 38 \\ & 2 \\ \hline 103\cdot 66 & 16 \\ \hline \end{array}$$

and as 7 added to 16 would not alter the 2nd decimal place, the result is 103·66.

26. The following kind of question is not unfrequently proposed, to be worked by arithmetic:

Find the interest on £573. 6*s*. 6*d*. for 213 days at $3\frac{1}{2}$ per cent. to pence.

As 6*s*. 6*d*. $= £\frac{6\frac{1}{2}}{20} = £\frac{3\frac{1}{4}}{10} = £\cdot 325$, the result may be put in the shape of finding, to three decimal places (which will give the farthings),

$$\frac{573\cdot 325 \times 7 \times 213}{73000}:$$

pointing the divisor after the 1st figure, this

$$=\frac{\cdot 0573325 \times 7 \times 213}{7\cdot 3},$$

and the division can *generally* be effected to 3 decimal places if the numerator be *so far* right; and to multiply the product *up to* 213 by 213, such product should be right to 6 decimal places to allow for 4 at the next step, thus:

```
 ·057332|5                7·|3|0|0|0 ) 85·482|(11·709
       7|                              73000|    20
 ·401327                               12482|   14·
     312                                7300|
 80·265|4                               5182|
  4 013|2                               5110|
  1 203|9                                 72| right
 85·482|5                                 65|
                                           7|
```

This is right, ∵ 73 goes 0 times into 72, ∴ the next figure is not > 9. Hence, as ·009 of £1 is less than $2\frac{1}{4}d.$ the interest is £11. 14*s.* 2*d.*

27. The more difficult cases in division are those in which the divisor is non-terminating, or at least the number of figures known is limited and only an approximation, as may also be the number known in the dividend. If the number known in the divisor be n, then, as previously shown, if the last figure be reserved for carrying, there will in general be only $n-2$ left for the contracted divisions, *i.e.* with safety only $n-2$ figures can be relied upon in the quotient (possibly $n-1$ if the 1st figure of the divisor be $> n$): always supposing that there are then enough in the dividend, *i.e.* as the division is by $n-1$ figures, there must be $n-1$ or n figures in the dividend, according to the number required for the 1st division; otherwise the quotient will probably be reduced.

For example (there being here no difficulty as to the dividend) the *modulus* of the common system of logarithms is given, in *Todhunter's Trigonometry*,

$$=\frac{1}{2{\cdot}30258509\ldots}={\cdot}43429448\ldots.$$

This appears to be regarded as a mere matter of ordinary division.

It might be expected here that only 7 figures could be relied upon from the division. Hence to divide by all the 9 figures in the divisor, taking the dividend to 9 decimal places and extending each multiplication row one place farther,

```
2·3|0|2|5|8|5|0|9 ) 1·000000000|0 ( ·43429448
                     921034036|a
                     ---------
                      78965964|
                      69077552|7 ?+2
                      --------
                       9888412|
                       9210340|3
                       -------
                        678072|
                        460517|0
                        ------
                        217555|
                        207232|6
                        ------
                         10323|
                          9210|3
                          ----
                          1113|
                           921|0
                           ---
                           192|
                           184|2
                           ---
                             8|
```

As allowance has to be made for possibly 8 subtractions and for a figure carried of from 0 to 3 to the 6 in the 1st multiplication row, ∴ the last figure of the quotient may be too great.

The 1st multiplication figure beyond the bar being unknown is represented by a, and the 2nd such figure may carry 2 more, therefore adding up the corresponding column, there are $a+7+3+0+6+3+0+2=21+a$ which equals 21 at the least, but the sum of the extra rows $< 21+a+8+2$, adding a possible 1 to each figure, or $< 31+a$. Thus on the whole the amount to be subtracted from 0 in the dividend, giving a its greatest value 9, is less than 40, and therefore 4 has to be carried within the bar at the most. This added to the possible 3 carried to the 6, makes 7 at most. Hence from 8 there is a remainder over and the result is correct.

In this the range for subtraction is already considerable from the introduction of unknown quantities, and the more there are of these the less likely is it that anything further that is definite can be obtained.

28. It appears usual to employ the *modulus* in the form just obtained, *i.e.*, to multiply the Napierian logarithm of a number by ·43429448, in order to get the ordinary logarithm. In practice, however, it would seem simpler to divide by 2·30258509.

Indeed, in general, abbreviated division is easier than multiplication, in cases where either may be chosen, without any additional operation. Thus $\log_e 3 = 1{\cdot}0986122\ldots$, and to get $\log_{10} 3$ to 7 decimal places safely it would probably be necessary in multiplying to know $\log_e 3$ to 9 decimal places, and the product would stand thus :

$$\begin{array}{r} 1{\cdot}098612288 \\ \underline{84492434}, \end{array}$$

and even then the last column might be insufficient for want of 1 or 2 more decimal places in the modulus. But by division to get safely 7 places, the 1st division should

be made by 8 figures, which in this case requires 8 decimal places of the dividend:

thus $2{\cdot}3025850(9)\,)\,1{\cdot}09861228\,($

gives the right answer, which may be easily worked from these figures, viz., ·4771212. In the ordinary tables the last figure is exaggerated into 3, in the old style.

29. In *Magnus's Mechanics* there is a question the answer to which in feet comes to

$$x=\frac{3300}{\sqrt{22379}};$$

as the figure under the root is nearly 150^2, the result is nearly 22, and is given in the book as "22·2 nearly." In such a case, the ordinary arithmetical method would be to rationalize the denominator, and (if the result be required to 1 decimal place)

$$x \text{ would thus equal } 3300\times\sqrt{22379}\div 22379.$$

As the quotient is to contain 3 figures the abbreviated division would be at the 1st step by 2237, requiring 4 figures in the dividend, correct and right to 'hundreds,' since to give 2 whole numbers in the quotient the dividend in full has 6. If then $\sqrt{22379}=149{\cdot}abc\ldots$, this would have to be multiplied by 3300 accurately to 'tens' at least, so that 2, if not 3, places of decimals are certainly required in the square root. This has been assumed to be $149{\cdot}abc$, because $150^2=22500$, and $150^2-149^2=1\times 299$, so that $149^2=22201$, which <22379.

By Division simply, involving 2 instead of 3 operations, to get 3 places in the quotient, the work would stand thus: $149{\cdot}a(b)\,)\,3300\,($, so that only 2 decimal

places are wanted in the square root for which, as above $22379 - 149^2 = 178$.

$\therefore$ for 2 more figures $2 \times 149 = 298$) 178·0 (·59
1490
290

$\therefore$ 14|9·|5 9)3300 (22·0
2991
309
299
10

The answer is, $\therefore$, 22·0 ft. to 1 decimal place.

30. To take an apparently very simple case, suppose $40\sqrt{10}$ required to 1 decimal place. Here no addition has to be allowed for, and yet it will be found necessary for certainty (per Art. 13) to get $\sqrt{10}$ to 4 decimal places ; for $\sqrt{10} = 3{\cdot}1622$, and

31·622
4
126·4 = result,

but without knowing the 4th decimal figure it would have been uncertain whether anything should or should not be carried to the 1st decimal figure in the result. For example, if the 4th decimal figure had been 5 instead of 2. there would have been 1 to carry.

31. Again, suppose $\frac{105}{24\pi}$ required to 1 decimal place. This $= \frac{35}{8\pi} = \frac{4{\cdot}375}{\pi}$, and carrying the dividend to 2 decimal places, as the 1st figure in $\pi = 3{\cdot}14\ldots$ is small, it follows that π is wanted to 2 decimal places only, 1 being the first figure in the quotient, thus

```
3·1|4 ) 4·37 ( 1·3
        314
        ---
        123
         94
        ---
         29 ?−2
```

Hence the result is 1·3.

It is obvious that the next figure in the quotient must be 8 or 9.

32. The result of the Target Problem in *Tod. Int. Cal.* ch. xiv., is $\frac{3\sqrt{3}}{4\pi}$: suppose this required to 1 decimal place. Since a division by 4 will not alter the accuracy of the decimal places, this may be done last, and $\frac{\sqrt{27}}{\pi}$ is wanted to 1 decimal place, $\therefore$ the *whole* figure is 1, $\therefore$ 2 figures are wanted in the quotient, $\therefore$ 3 in the divisor, $\therefore$ 3 in the dividend, and $\therefore$ $\sqrt{27}$ must be obtained to 2 decimal places. Now $\sqrt{27{\cdot}04} = 5{\cdot}2$, so that the root is so nearly 5·2 that it may be assumed $= 5{\cdot}19$.

[In fact $(5{\cdot}2)^2 = (5{\cdot}19 + {\cdot}01)^2$, $\therefore$ $27{\cdot}04 > (5{\cdot}19)^2 + {\cdot}1$, $\because$ $2 \times 5{\cdot}19 \times {\cdot}01 = {\cdot}1038$, and $\therefore$ $27 > (5{\cdot}19)^2$.]

Thus,

```
3·1|4 ) 5·19| ( 1·6
        314 |
        ----|
        205 |
        188 |
        ----|
         17 | ?−2
```

Hence the result required $= \frac{1{\cdot}6}{4} = {\cdot}4$.

As the next figure in the quotient to 1·6 will clearly be 4 or 5, the result to 2 decimal places $= {\cdot}41$.

33. Here is a little question in elementary dynamics: "To find g to 2 decimal places in feet *or* inches, when $g = \pi^2 \times l$, and $l = 39{\cdot}1386$ inches."

Trying for 3 decimal places, if $\pi^2 = 9{\cdot}abc\ldots$ the product would stand thus :

$$\begin{array}{r} 39{\cdot}1386 \\ \underline{edcba9\ ,} \end{array}$$

$\therefore$ 5 places of decimals are wanted in π^2. Trying then 7 places,

$$\begin{array}{rl} 3{\cdot}1415926 & \\ \underline{62951413} & \\ 94247778 & ?+2 \\ 3141592 & \\ 1256637 & \\ 31415 & \\ 15707 & \\ 2827 & \\ 62 & \\ \underline{18} & ?+2 \\ 9{\cdot}86960|36 & \end{array}$$

as the additions to be made, of 8 for the rows, 2 for the figure to be possibly carried at the 1st multiplication, and of two more for the 8th (unknown) decimal figure in π, which might be 9, would make no difference in the above to 5 decimal places, it is so far correct. Next,

$$\begin{array}{rl} 39{\cdot}1386 & \\ \underline{069689} & \\ 35224|7 & ?+1 \\ 3131|0 & \\ 234|8 & \\ 35|2 & \\ \underline{2|3} & \\ 386{\cdot}28|0 & ?+1 \end{array}$$

The $?+1$ is introduced in case the unknown figures (if any) in l should be $> \dot{6}$; and the result is right to 2 deci-

mal places, because 5 or 6 added to the barred column would not alter them.

Hence $g = 386{\cdot}28$ ins. $= 32{\cdot}19$ feet.

The example is only given to show a way to approximate in such cases by arithmetic, as it is easily worked by logarithms.

If g were required to be really accurate to 4 decimal places, the question might have to be worked by arithmetic.

METHOD FOR SQUARING A DECIMAL QUANTITY.

34. In algebra it is shown that the square of the sum of any quantities = the sum of their squares and of twice the product of every two of them. This may be easily and neatly applied to decimal quantities as follows :—To take a definite case, suppose π^2 wanted to say 5 decimal places, π being given to 7, *i.e.* $= 3{\cdot}1415926$, to allow for the additions which will arise.

Here each digit may be regarded as a separate number with its proper value, and the square of $3 = 9$, that of $\cdot 1 = \cdot 01$, that of $\cdot 04 = \cdot 0016$, and so on, so that the squares will fit at once into one row, and thus be very easily written down, as the square of no digit can contain more than 2 figures. This row is $\therefore$ 9·01,16,01,25,81,04,36.

Now, for the products, 2×3 multiplied by each of the other figures amounts to 6 times the figures after the 1st or $6 \times \cdot 1415926$; then $2 \times \cdot 1$ times the figures which are beyond the 2nd figure, amount to $\cdot 2 \times \cdot 0415926$; $2 \times \cdot 04$ times the figures beyond the 2nd decimal place amount to $\cdot 08 \times \cdot 0015926$, and so on.

$$\begin{array}{rl} \text{Hence } \pi^2 = & 9{\cdot}0116012 \\ & +{\cdot}8495556\ (?+5 \text{ carried}) \\ & +{\cdot}0083185 \\ & +{\cdot}0001274 \\ & +{\cdot}0000011 \\ \hline & 9{\cdot}86960|38 \end{array}$$

The rows end here to 7 decimal places, and there are 5 of them, and an addition of $5+5$ (possibly carried to the 2nd row) will not alter the 5th decimal place.

$$\text{Hence } \pi^2 = 9{\cdot}86960 \text{ so far.}$$

It will be observed that for the first product the 7th decimal place in π is started with ; for the 2nd product the 6th decimal place, and so on : and if the significant figures be properly placed, the noughts in the last 3 addition rows may be dispensed with.

35. The following question occurs in one of the old books :—" A banker borrows money at £3. 10s. per cent. per annum interest, payable at the end of the year ; and lets it out at £5 per cent. per annum interest, payable quarterly, and by this means gains £200 a year ; how much does he borrow ?"

The answer may be put in the form

$$£200 \div \{(1{\cdot}0125)^4 - 1{\cdot}035\}.$$

I. *By Logarithms :*

$$\begin{array}{rr} \log 1{\cdot}0125 = & 0{\cdot}005395 \\ & 4 \\ \hline & 0{\cdot}0215800 \\ & 614 \\ \hline & 186 \\ & 166 \\ \hline & 20 \\ \hline \end{array}$$

$$\therefore \quad (1{\cdot}0125)^4 = \begin{array}{r} 1{\cdot}050944 \\ -1{\cdot}035 \\ \hline 0{\cdot}015944 \end{array}$$

and

$$\begin{array}{rl} \log 200 = & 2{\cdot}3010300 \\ \log 0{\cdot}015944 = & \bar{2}{\cdot}2025973 \\ \hline & 4{\cdot}0984327 \\ & \underline{014} \\ & \underline{313} \end{array}$$

Hence the amount is £12543·9.

II. *By Arithmetic:* As the quarterly interest is not much more than 5 per cent. yearly, ∴ roughly

$$(1{\cdot}0125)^4 = 1{\cdot}05,$$

so that the denominator to be found here has its 1st significant figure in the 2nd decimal place and < 2, ∴ the numerator will give whole figures to 5 places, and to get these, 6 significant figures are wanted in the denominator, ∴ $(1{\cdot}0125)^4$ should be right to 7 decimal places.

Now

$$(1{\cdot}0125)^2 = \begin{array}{r} 1{\cdot}00010425 \\ +\ {\cdot}025{,}050 \\ +\ \underline{2} \\ \hline 1{\cdot}02515625 \end{array}$$

and, trying to 8 decimal places,

$$(1{\cdot}02{,}51{,}56{,}25)^2 = \begin{array}{r} 1{\cdot}00042501 \\ +\ {\cdot}05031250 \\ 20625 \\ 156 \\ \underline{1} \\ \hline 1{\cdot}0509453|3 \end{array}$$

and as 5 added to the last figure would not alter the 7th

decimal place, the result is so far true, and ∴ the denominator is ·0159453, and thus

```
1·5|9|4|5|3... )20000·0| ( 12542
                159453|
                 40546| right
                 31890|6
                  8656|
                  7972|6
                   684|
                   637|
                    47| −1
                    31|
```

At the 1st subtraction the dividend continuing with 0's, 1 must be carried ; and the *most* that could be left in the 1st remainder beyond the bar is 9, but the 2nd and 3rd multiplication rows continued 1 step make up at least 12, ∴ at least 1 more must be carried, making the last remainder at most 46 instead of 47, and as $3 \times 15|9...$ would make $47|...$, ∴ the last figure in the quotient is not quite 3.

Hence the logarithmic result is wrong to pounds, and not even correct beyond 4 significant figures. This is only given to show that logarithms require care with a view to the degree of accuracy wanted, for in the above $(1{\cdot}025)^4$ cannot be obtained to 8 significant figures by logarithms : the figures should have been transformed.

EXAMPLES FOR EXERCISE.

1. Add together to 2 decimal places 376·25, 86·125, 637·4725, 6·54, 358·865, and 41·02. *Result:* 1506·27.

2. Add together to whole numbers 3·5, 47·25, 927·01, 2·0073, and 1·5. *Result:* 981.

3. Add together to 1 decimal place 276, 54·321, ·65, 112, 1·25, and ·0463. *Result:* 444·2.

4. Find the sum to 3 decimal places of $67{\cdot}3\dot{4}\dot{5}$, $8{\cdot}\dot{6}2\dot{1}$, $\cdot\dot{2}\dot{4}$, and $\cdot\dot{8}$. *Result:* 77·098.

5. Find the sum to 2 decimal places of ·5, $\cdot\dot{2}\dot{5}$, $17{\cdot}4\dot{7}$, $9{\cdot}\dot{6}5\dot{1}$, and $67{\cdot}3\dot{4}\dot{5}$. *Result:* 95·22.

6. Find the sum to 2 decimal places of $9{\cdot}\dot{8}1\dot{4}$, 1·5, $87{\cdot}2\dot{6}$, $0{\cdot}8\dot{3}$, and $124{\cdot}0\dot{9}$. *Result:* 223·51.

7. Find the previous sum to 4 decimal places. *Result:* 223·5148.

8. Add together to 3 decimal places, 5391·357, $72{\cdot}3\dot{8}$, $187{\cdot}2\dot{1}$, $4{\cdot}296\dot{5}$, $217{\cdot}849\dot{6}$, $42{\cdot}17\dot{6}$, $\cdot52\dot{3}$, and $58{\cdot}3004\dot{8}$. *Result:* 5974·103.

9. Add to 5 decimal places $\cdot\dot{1}6\dot{2}$, $134{\cdot}\dot{0}\dot{9}$, $2{\cdot}\dot{9}\dot{3}$, $97{\cdot}\dot{2}\dot{6}$, 3·7692303, $99{\cdot}08\dot{3}$, 1·5, and $\cdot\dot{8}1\dot{4}$. *Result:* 339·62246.

10. From Art. 8, find (1) 1, (2) 2, (3) 3 more decimal places in *e*. *Result:* The next 3 places are 459.

11. Take 13·725 from 127·62 to 1 decimal place. *Result:* 113·8.

12. From $\cdot4\dot{2}\dot{7}$ take $\cdot0\dot{3}\dot{4}$ to 4 decimal places. *Result:* ·3929.

13. From $3{\cdot}85\dot{6}$ take $\cdot03\dot{8}$ to 5 decimal places. *Result:* 3·81777.

14. Find the sum of £$\frac{2}{25}$, £$\frac{1}{800}$, and £$\frac{1}{128}$ accurately to pence. *Result:* 1*s.* 9*d.*

15. Find, to pence, $£\frac{1}{3}+£\frac{1}{5}+£\frac{1}{7}+£\frac{1}{11}+£\frac{1}{13}+£\frac{1}{17}$.

Result: 18*s.* 0*d.*

16. Given $\pi=3{\cdot}14159265358979 3\ldots$, subtract $\frac{1}{4}({\cdot}00005)^3$ from $\frac{\pi}{64800}$, to 15 decimal places.

Result: ·000048481368079.

17. Find $\frac{1}{\sqrt{3}}$ to 7 places of decimals. *Result:* ·5773502.

18. Find $\frac{16}{\sqrt{3}}$ to 3 decimal places. *Result:* 9·237.

19. Obtain to 2 decimal places $6-\frac{12}{\sqrt{7}}$. *Result:* 1·46.

20. To subtract a radical from a greater whole number correctly to 2 decimal places, to how many should the radical be calculated ?

21. Multiply 79·347 by 23·15 to whole figures.

Result: 1836.

22. Multiply to 2 decimal places ·63478 by ·8204.

Result: ·52.

23. Multiply to 4 decimal places, ·02534 by ·03256.

Result: ·0008.

24. Find the product $e\pi$ to 3 decimal places.

Result: 8·539.

25. Divide 14 by ·7854 to 1 decimal place. *Result:* 17·8.

26. Divide 234·70525 by 64·25 to 1 decimal. *Result:* 3·6.

27. Divide ·48624096 by 179 to 4 decimal places.

Result: ·0027.

28. Find $\frac{40}{\sqrt{3}}$ to 2 decimal places. *Result:* 23·09.

29. Obtain $\frac{20}{\sqrt{221}}$ to 2 decimal places. *Result:* 1·34.

30. Find $\frac{1}{e}$ to 5 decimal places. *Result:* ·36787.

31. Find e^2 to 5 decimal places. *Result:* 7·38905.

32. Find $\frac{1}{e^2}$ to 5 decimal places. *Result:* ·13533.

33. Obtain $(3{\cdot}777)^2$ to 2 decimal places. *Result:* 14·26.

34. If 10000565·278 French metres = 32810846·2868 English feet, how many inches are there in 1 metre, in whole numbers? To how many decimal places could this answer be extended if the above figures be accurate?

Result: (1) 39, (2) 10.

35. What is the velocity of a point describing uniformly $\pi \times 41847000$ ft. in 86,164 seconds, (1) to whole numbers, (2) to 2 decimal places. *Results:* 1525·76.

36. If V be the velocity in the previous example, show that $\frac{V^2}{\frac{1}{2} \times 41847000} = 0{\cdot}11126$. $\left[\text{Reduce to } \frac{\pi^2 \times {\cdot}83694}{(8{\cdot}6164)^2}\right]$

37. If $g = 32{\cdot}088$, what fraction of g is the last result?

Ans.: $\frac{1}{288{\cdot}4}$.

38. If $g = \pi^2 l$, and $12l = 39{\cdot}139830$, find g to 4 decimal places. *Result:* 32·1912.

39. If any finite number of the terms giving e be added together, will their sum be commensurable or not?

40. Obtain $(86{\cdot}14)^2$ to whole numbers. *Result:* 7420.

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